straint torques, result:

$$\begin{bmatrix} a_{00} & \underset{\sim}{a}_{01} & \underset{\sim}{a}_{02}\cdots\underset{\sim}{a}_{0,r-3} \\ \underset{\sim}{a}_{10} & & \\ \vdots & & \\ \underset{\sim}{a}_{r-3,0} & & \end{bmatrix} \begin{bmatrix} \dot{\underset{\sim}{\omega}}_0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \\ \vdots \\ \ddot{\gamma}_{r-3} \end{bmatrix} =$$

$$\begin{bmatrix} \sum_\lambda \underset{\sim}{E}^*_\lambda - \sum_\lambda \sum_{\mu\neq\lambda} m_\mu \underset{\sim}{\mathfrak{L}}_{\lambda\mu} \times [\ddot{\underset{\sim}{D}}^R_{\lambda\mu} + 2\underset{\sim}{\omega}_\lambda \times \dot{\underset{\sim}{D}}^R_{\lambda\mu}] \\ + \sum_\mu \sum_{\mu\neq\lambda} \underset{\sim}{D}_{\lambda\mu} \times m[\ddot{\underset{\sim}{D}}^R_{\mu\lambda} + 2\underset{\sim}{\omega}_\mu \times \dot{\underset{\sim}{D}}^R_{\mu\lambda}] \\ \hat{\underset{\sim}{g}}_1 \cdot \Big\{ \sum_\lambda \epsilon_{1\lambda} \underset{\sim}{E}^*_\lambda - \sum_\lambda \epsilon_{1\lambda} \sum_{\mu\neq\lambda} m_\mu \underset{\sim}{\mathfrak{L}}_{\lambda\mu} \times [\ddot{\underset{\sim}{D}}^R_{\lambda\mu} + 2\underset{\sim}{\omega}_\lambda \times \dot{\underset{\sim}{D}}^R_{\lambda\mu}] \\ + \sum_\lambda \epsilon_{1\lambda} \sum_{\mu\neq\lambda} \underset{\sim}{D}_{\lambda\mu} \times m[\ddot{\underset{\sim}{D}}^R_{\mu\lambda} + 2\underset{\sim}{\omega}_\mu \times \dot{\underset{\sim}{D}}^R_{\mu\lambda}] \Big\} \\ \vdots \\ \hat{\underset{\sim}{g}}_{r-3} \cdot \Big\{ \sum_\lambda \epsilon_{r-3,\lambda} \underset{\sim}{E}^*_\lambda - \sum_\lambda \epsilon_{r-3,\lambda} \sum_{\mu\neq\lambda} m_\mu \underset{\sim}{\mathfrak{L}}_{\lambda\mu} \times [\ddot{\underset{\sim}{D}}^R_{\lambda\mu} + 2\underset{\sim}{\omega}_\lambda \\ \times \dot{\underset{\sim}{D}}^R_{\lambda\mu}] + \sum_\lambda \epsilon_{r-3,\lambda} \sum_{\mu\neq\lambda} \underset{\sim}{D}_{\lambda\mu} \times m[\ddot{\underset{\sim}{D}}^R_{\mu\lambda} + 2\underset{\sim}{\omega}_\mu \times \dot{\underset{\sim}{D}}^R_{\mu\lambda}] \Big\} \end{bmatrix}$$

where

$$a_{00} = \sum_\lambda \sum_\mu \Phi_{\lambda\mu}, \text{ a dyadic}$$

$$a_{0k} = \sum_\lambda \sum_\mu \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot \hat{\underset{\sim}{g}}_k, \text{ a vector}$$

$$\underset{\sim}{a}_{i0} = \hat{\underset{\sim}{g}}_i \cdot \sum_\lambda \sum_\mu \epsilon_{i\lambda} \Phi_{\lambda\mu}, \text{ a vector}$$

$$a_{ik} = \hat{\underset{\sim}{g}}_i \cdot \sum_\lambda \sum_\mu \epsilon_{i\lambda} \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot \hat{\underset{\sim}{g}}_k, \text{ a scalar}$$

$$\underset{\sim}{E}^*_\lambda = \underset{\sim}{E}_\lambda - \sum_\mu \Phi_{\lambda\mu} \cdot \sum_k \epsilon_{k\mu} \dot{\gamma}_k \hat{\underset{\sim}{g}}_k$$

and

$\epsilon_{i\mu} = 1$, if $\hat{\underset{\sim}{g}}_i$ belongs to a joint anywhere on the chain of bodies connecting body μ and the reference body
$= 0$, otherwise (e.g., if $\mu = 0$)

The use and numerical integration of Eqs. (16) will be little changed from that of the original equations, except that the joint velocities and accelerations with respect to the bodies they connect, $\dot{\underset{\sim}{\mathfrak{L}}}^R_{\lambda\mu}$ and $\ddot{\underset{\sim}{\mathfrak{L}}}^R_{\lambda\mu}$, respectively, must be prescribed and hence $\dot{\underset{\sim}{D}}^R_{\lambda\mu}$ and $\ddot{\underset{\sim}{D}}^R_{\lambda\mu}$ determined from Eq. (5). To determine the spacecraft attitude with respect to an external frame, a set of first-order equations, such as Euler angle rate equations, must be added. We have found that as a check on the accuracy of the computer code, which can be lengthy, it is helpful to add a routine which determines total system angular momentum about the mass center and check for its conservation when no external torques are considered.

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Analytic Fourier Transform for a Class of Finite-Time Control Problems

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Introduction

A COMMON method for investigating the effectiveness of various control designs consists of studying frequency domain characteristics of the control, by numerically evaluating the required Fourier transform. For finite-time open- and closed-loop control problems, this can be accomplished by either numerically integrating the integral definition of Fourier transform for each frequency of interest, or using a fast Fourier transform algorithm. Alternatively, we present in this Note a computationally efficient closed-form solution for the Fourier transform of finite-time open- and closed-loop control problems, where the dynamics of the control is governed by matrix exponentials.

Problem Formulation

The fundamental definition of the complex Fourier transform follows as

$$\bar{u}(\omega) = \int_0^T u(t) e^{-i\omega t} \mathrm{d}t \qquad (n \times 1) \tag{1}$$

where $u(t)$ is assumed to be given by[1-4]

$$u(t) = A e^{Bt} b \qquad (n \times 1) \tag{2}$$

where A is $n \times m$, B is $m \times m$, $e^{(\cdot)}$ is the matrix exponential, and b is $m \times 1$.

Introducing Eq. (2) into Eq. (1) yields

$$\bar{u}(\omega) = A\xi(\omega) \tag{3}$$

where

$$\xi(\omega) = \int_0^T e^{Bt} b e^{-i\omega t} \mathrm{d}t \tag{4}$$

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As shown in Ref. (5), the integral appearing in Eq. (4) can be evaluated by defining the complex matrix

$$D(\omega)=\underbrace{\begin{bmatrix} -B & b \\ 0 & -i\omega \end{bmatrix}}_{n \quad\quad 1}\begin{matrix} \}n \\ \}1 \end{matrix} \tag{5}$$

and computing the matrix exponential

$$e^{D\tau}=\begin{bmatrix} F_1 & G_1 \\ 0 & F_2 \end{bmatrix}=\begin{bmatrix} e^{-B\tau} & e^{-B\tau}\xi(\omega) \\ 0 & e^{-i\omega\tau} \end{bmatrix} \tag{6}$$

from which it follows that

$$\xi(\omega)=e^{B\tau}G_1(\omega) \tag{7}$$

Since the numerical effort required to compute G_1 for each desired value of ω is prohibitive, we present in the next section a coordinate transformation that greatly reduces the computational burden required to produce $\xi(\omega)$ in Eq. (7).

Reducing Subspace Coordinate Transformation

In this section we present an algorithm for reducing the complex matrix D in Eq. (5) to block diagonal form by a similarity transformation.[6] In particular, we see a complex nonsingular matrix Φ, such that $\Phi^{-1}D\Phi$ has the form

$$\Phi^{-1}D\Phi=\tilde{D}=\mathrm{diag}(-B,\ -i\omega) \tag{8}$$

The transformation matrix Φ is assumed to have the special form

$$\Phi=\begin{bmatrix} I & -p \\ 0 & 1 \end{bmatrix}$$

where the inverse of Φ can be shown to be

$$\Phi^{-1}=\begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

Hence,

$$\Phi^{-1}D\Phi=\begin{bmatrix} -B & Bp-i\omega p+b \\ 0 & i\omega \end{bmatrix} \tag{9}$$

and the problem of determining Φ becomes that of solving the following linear equation for p:

$$[B-i\omega I]p(\omega)=-b \tag{10}$$

where the solution for p is well defined provided that $i\omega$ is not an eigenvalue of B.

Table 1 Solution techniques for $p(\omega)$

B Diagonalizable	B Ill-conditioned eigensystem
$\Lambda=\mathrm{diag}\ [\lambda_1,\ ...,\ \lambda_n]$	$\Lambda=$upper quasitriangular[a]
$\beta=L^Tb$	$\beta=U^Tb$
$\gamma_\pi=-\beta_j/(\lambda_j-i\omega)\ (j=1,...,n)$	$[\Lambda-i\omega I]\gamma j=-\beta$ (easy to solve)
$p(\omega)=R\gamma(\omega)$	$p(\omega)=U\gamma(\omega)$
$R=$right eigenvector of B	$U^TBU=\Lambda$ (real Schur decomposition)
$L^TBR=\Lambda$	$U^TU=I$ (orthogonality)
$L=$left eigenvector of B	
$L^TR=I$ (biorthogonality)	

[a] A quasitriangular matrix is triangular with possible nonzero 2×2 blocks on the diagonal.

From Eq. (8) it follows that the matrix exponential of Eq. (6) can be written as

$$e^{D\tau}=\Phi e^{\tilde{D}\tau}\Phi^{-1}=\begin{bmatrix} e^{-B\tau} & e^{-B\tau}p-pe^{-i\omega\tau} \\ 0 & e^{-i\omega\tau} \end{bmatrix} \tag{11}$$

Comparing Eqs. (6) and (11) it follows that the desired integral for $\xi(\omega)$ in Eq. (7) is given by

$$\xi(\omega)=p-e^{B\tau}pe^{-i\omega\tau} \tag{12}$$

where the entire solution follows after determining p from Eq. (10) for each frequency of interest. The significant feature of Eq. (12) is that the computationally intensive solution for $e^{B\tau}$ must be carried out only one time, thus greatly reducing the labor required to produce $\xi(\omega)$.

Solution for the Uncoupling Transformation Vector

Since the B matrix in Eq. (10) is constant and generally fully populated, we seek a solution technique that minimizes the computational effort. However, we recognize that there are two classes of solutions possible: 1) systems where B is diagonalizable; and 2) systems where the eigensystem for B is ill-conditioned.

The solutions for both classes of problems are obtained via a "transformation method." In particular, such methods are based upon the equivalence of the problems[7,8]

$$[B-i\omega I]p(\omega)=-b, \qquad [\Lambda-i\omega I]\gamma(\omega)=-\beta \tag{13}$$

The solution algorithms for both classes of problems are listed in Table 1. However, if $i\omega=\lambda_j$, then Eq. (6) can be used to obtain the solution. The desired solution for $\bar{u}(\omega)$ follows on introducing Eq. (12) into Eq. (3), yielding

$$\bar{u}(\omega)=A\{p-e^{B\tau}pe^{-i\omega\tau}\} \tag{14}$$

In order to efficiently evaluate Eq. (14), it is necessary to recast the equation in the form

$$\bar{u}(\omega)=A_1\gamma(\omega)-A_2\gamma(\omega)e^{-i\omega\tau} \tag{15}$$

where $A_1=AR$ and $A_2=Ae^{B\tau}R$ if B is diagonalizable, and $A_1=AU$ and $A_2=Ae^{B\tau}U$ if B is ill-conditioned.

Example Application

Given the first-order system

$$\dot{x}=-x+u;\quad \text{given } x(0)=0,\quad x(\tau)=1$$

we seek the control u to minimize

$$J=\frac{1}{2}\int_0^\tau u^2\mathrm{d}t$$

The open-loop control can be shown to be

$$u(t)=-\lambda(t) \qquad (\lambda=\text{costate}) \tag{16}$$

where

$$\begin{Bmatrix} x(t) \\ \lambda(t) \end{Bmatrix}=e^{Bt}\begin{Bmatrix} x(0) \\ \lambda(0) \end{Bmatrix},\ B=\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \tag{17}$$

$$e^{Bt}=\begin{bmatrix} e^{-t} & -\sinh t \\ 0 & e^t \end{bmatrix};\quad \lambda(0)=-1/\sinh\tau \tag{18}$$

Thus the control of Eq. (16) can be written as

$$u(t) = Ae^{Bt}b = e^t/\sinh\tau \tag{19}$$

where $A = [0 \quad -1]$, e^{Bt} is defined by Eq. (18), and $b = (0_g - 1/\sinh\tau)^T$. The analytic Fourier transform of $u(t)$ follows as

$$\bar{u}(\omega) = \left(\int_0^\tau e^{(1-i\omega)t}\mathrm{d}t\right)/\sinh\tau = [e^{(1-i\omega)\tau} - 1]/[(1-i\omega)\sinh\tau] \tag{20}$$

Since B is diagonalizable, we use the right and left eigenvector transformation method to solve Eq. (13), leading to

$$R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 2/\sqrt{2} \end{bmatrix}, \quad L = \begin{bmatrix} 2/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

from which it follows that

$$\beta = L^T b = -[1/(\sqrt{2}\sinh\tau), 1/(\sqrt{2}\sinh\tau)]^T$$

$$\gamma = (-1/[(1+i\omega)\sqrt{2}\sinh\tau], 1/[(1-i\omega)\sqrt{2}\sinh\tau])^T$$

$$A_1 = AR = [0, -2/\sqrt{2}], \quad A_2 = Ae^{B\tau}R = [0, -2e^\tau/\sqrt{2}]$$

Thus, from Eq. (15) we have

$$\bar{u}(\omega) = [e^{(1-i\omega)\tau} - 1]/[(1-i\omega)\sinh\tau] \tag{21}$$

where Eq. (21) agrees with Eq. (20).

Conclusions

A computationally efficient algorithm has been presented for obtaining the complex Fourier transform of a class of vector functions that frequently occur in modern control theory. The basic algorithm requires 1) the evaluation of a single matrix exponential for the dynamics of the time-varying control; 2) the solution for either the right and left eigenvectors or a real Schur decomposition of the constant control dynamics matrix; 3) the sequential solution for the reducing subspace transformation vector p; and 4) the evaluation of a single scalar complex exponential.

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A Robust Compensator Design by Frequency-Shaped Estimation

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Introduction

MUCH has been made in recent literature[1-4] regarding the lack of robustness of servo loops designed by so-called 'optimal' stochastic control theory, i.e. Linear-Quadratic Synthesis (LQS) methods. It has even been claimed[5] that LQS should not be used on control systems for aerospace vehicles because of its sensitivity to plant uncertainties and its inappropriate bandwidth characteristics.

Frequency domain analysis of multivariable control loops indicates that linear state feedback provides good stability margins,[6] but that estimated state feedback can reduce these margins to zero.[1] It has been proposed to modify the Linear-Quadratic-Gaussian (LQG) estimator design procedure by adding fictitious process noise at the control inputs to restore the margins,[2,7] but that approach typically introduces high frequency modes into the estimator as the spectral density of this noise is increased.

This note describes an example using a frequency-shaped cost functional on measurement noise in LQG estimator design as a means for improving control loop robustness. The theory and implementation of frequency-shaped LQS design has been described elsewhere.[8-11] Kim[9,10] has shown that shaping the measurement noise of individual measurements using classical compensators, such as lead or lag networks, as shaping filters is effective in modifying multivariable controller robustness. This procedure results in transmission zeros being inserted into the estimator transfer matrix at prescribed frequencies. The selection of these frequencies (i.e. of the shaping filters) is accomplished by a frequency response analysis of the standard LQS controller design. The resulting estimator has higher order than the 'optimal' estimator, but it is no more complicated to design, and its bandwidth can be maintained in a reasonable range by choice of the usual LQS weighting parameters.

A Design Application

As an illustrative example of the effect of this procedure on frequency response of a multivariable loop, a controller design was undertaken on the single-input/two-output, 4th-order model for the azimuth pointing control servo of the Multiple Mirror Telescope.[12] This optical telescope, sponsored by the Smithsonian Institute and the University of Arizona, consists of six coaligned 1.8-meter primary mirrors in a hexagonal structure on an azimuth-altitude mount. The pointing controls include DC electric motors driving ring gears through a 100-to-1 reduction. Compliance in the gears combines with motor dynamics to produce the 4th-order model. A classically designed compensator with a bandwidth of about .5 Hz is presently used for control on each axis.

A state space representation of the azimuth axis system in the form

$$\dot{x} = Ax + Bu + Gw, \qquad z = Cx + v \tag{1}$$

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